FUZZY LOGIC
**Logic**

\Log"ic\, n. 1. The science or art of exact reasoning, or of pure and formal thought, or of the laws according to which the processes of pure thinking should be conducted; the science of the formation and application of general notions; the science of generalization, judgment, classification, reasoning, and systematic arrangement; correct reasoning.


**logic**

n 1: the branch of philosophy that analyzes inference 2: reasoned and reasonable judgment; "it made a certain kind of logic" 3: the principles that guide reasoning within a given field or situation; "economic logic requires it"; "by the logic of war" 4: a system of reasoning [syn: logical system, system of logic]

*Source*: *WordNet ® 1.6, © 1997 Princeton University*
INFERENCE

inference
\In"fer*ence\, n. [From Infer.]

1. The act or process of inferring by deduction or induction.

2. That which inferred; a truth or proposition drawn from another which is admitted or supposed to be true; a conclusion; a deduction. -Milton.

Usage: Inference, Conclusion. An inference is literally that which is brought in; and hence, a deduction or induction from premises, something which follows as certainly or probably true. ``An inference is a proposition which is perceived to be true, because of its connection with some known fact."``When something is simply affirmed to be true, it is called a proposition; after it has been found to be true by several reasons or arguments, it is called a conclusion."--I. Taylor.

### Classical Proposition Logic

As in our ordinary informal language, “sentence” is used in the logic. Especially, a sentence having only “true (1)” or “false (0)” as its truth value is called “proposition”.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Truth Value</th>
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<tbody>
<tr>
<td>2 + 4 = 7</td>
<td>false</td>
</tr>
<tr>
<td>For every $x$, if $f(x) = \sin x$, then $f'(x) = \cos x$.</td>
<td>true</td>
</tr>
<tr>
<td>It rains now.</td>
<td>true or false depending whether it rains or not</td>
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</tbody>
</table>
CLASSICAL PROPOSITION LOGIC

The followings are not propositions

- Why are you coming to class?
- He hits 5 home runs in one season.
- $x + 5 = 0$
- $x + y = z$
CLASSICAL PROPOSITION LOGIC

Logic variable

- If we represent a proposition as a variable, the variable can have the value true or false.
- This type of variable is called as a “proposition variable” or “logic variable”
CLASSICAL PROPOSITION LOGIC

- Connectives - combine prepositional variables
  - Negation $\overline{a}$
  - Conjunction $a \land b$
  - Disjunction $a \lor b$
  - Implication $a \rightarrow b$
  - etc.

| $p$ | $q$ | $\neg p$ | $p \lor q$ | $p \land q$ | $p \rightarrow q$
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**Logical Implication**

- What does it mean if $p$ implies $q$?

- Example:
  - $p = \text{sky is overcast}$,
  - $q = \text{sun not visible}$,
  - $p \rightarrow q$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
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http://if.kaist.ac.kr/lecture/cs670/textbook/
Logic functions

- Logic function
  - a combination of propositional variables by using connectives

- Logic formula
  - Truth values 0 and 1 are logic formulas
  - If \( v \) is a logic variable, \( v \) and \( v' \) are a logic formulas
  - If \( a \) and \( b \) represent a logic formulas, \( a \land b \) and \( a \lor b \) are also logic formulas

http://if.kaist.ac.kr/lecture/cs670/textbook/
Tautology and Inference Rule

Tautology

- A “tautology” is a logic formula whose value is always true regardless of its logic variables.
- A “contradiction” is one which is always false.

Tautology \((a \rightarrow b) \rightarrow \overline{b}\)
TAUTOLOGY AND INference RULE

\[(a \land (a \rightarrow b)) \rightarrow b\]

This tautology means that
  “If a is true and (a → b) is true, then b is true.”
or  “If a exists and the relation (a → b) is true,
then b exists.”

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TAUTOLOGY AND INFRINGEMENT RULE

Important inference rules that use tautologies:

- **Modus Ponens**
  \[(a \land (a \rightarrow b)) \rightarrow b\]

- **Modus Tollens**
  \[(\sim b \land (a \rightarrow b)) \rightarrow \sim a\]

- **Hypothetical syllogism**
  \[((a \rightarrow b) \land (b \rightarrow c)) \rightarrow (a \rightarrow c)\]
Predicate logic

“Predicate logic” is a logic which represents a proposition with the predicate and an individual (object)

“Socrates is a man”
- “Socrates” – object
- “is a man” – predicate

“Two is less than four”
- “Two”, “four” – objects
- “is less than” - predicate
**Predicate Logic**

Objects in predicate logic can be represented by variables. Then a predicate proposition can be evaluated for truth if an element of a universal set is instantiated to the variable.

“*x is a man*”
- *x* = “Tom”, the proposition becomes “Tom is a man”

“*x satisfies P*” can be denoted *P(x)*
- *is_a_man*(Tom)
QUANTIFIERS

- **Universal quantifier**
  - “for all”
  - Denoted symbolically by \( \forall \)

- **Existential quantifier**
  - “there exists”
  - Denoted symbolically by \( \exists \)
In a fuzzy expression (formula), a fuzzy proposition can have its truth value in the interval $[0,1]$

$$f : [0,1] \rightarrow [0,1]$$

Generalization to $n$ dimensions

$$f : [0,1]^n \rightarrow [0,1]$$
A Fuzzy Proposition:
Not every book on fuzzy logic is a book on fuzzy logic!
Fuzzy Logic

- **Definition of Fuzzy logic**

  Fuzzy logic is a logic represented by a fuzzy expression (formula) which satisfies the following:

  - Truth values, 0 and 1, and variables \( x_i \) (\( \in [0,1] \), \( i = 1, 2, ..., n \)) are fuzzy expressions
  - If \( f \) is a fuzzy expression, \(~f\) (not \( f\)) is also a fuzzy expression
  - If \( f \) and \( g \) are fuzzy expressions, \( f \land g \) and \( f \lor g \) are also fuzzy expressions
Operators in fuzzy expressions

- **Negation** $a' = 1 - a$

- **Conjunction** $a \land b = \min (a, b)$

- **Disjunction** $a \lor b = \max (a, b)$

- **Implication** $a \rightarrow b = \min (1, 1+b-a)$
Apparatus of fuzzy logic is built on:

- Fuzzy sets: describe the value of variables
- Possibility distributions: constraints on the value of a variable

⇒ Linguistic variables: qualitatively and quantitatively described by fuzzy sets

Fuzzy if-then rules: knowledge
LINGUISTIC VARIABLES

- **Linguistic variable** is "a variable whose values are words or sentences in a natural or artificial language". Each linguistic variable may be assigned one or more linguistic values, which are in turn connected to a numeric value through the mechanism of membership functions.

- **Motivation**: Conventional techniques for system analysis are intrinsically unsuited for dealing with systems based on human judgment, perception & emotion
Principle of incompatibility: As the complexity of a system increases, our ability to make precise & yet significant statements about its behavior decreases until a fixed threshold. Beyond this threshold, precision & significance become almost mutually exclusive characteristics [Zadeh, 1973]

LV represented by a quintuple \((x, T(x), U, G, M)\)
- \(x\): name of variable
- \(T(x)\): set of linguistic terms which can be a value of the variable
- \(U\): set of universe of discourse which defines the characteristics of the variable
- \(G\): syntactic grammar which produces terms in \(T(x)\)
- \(M\): semantic rules which map terms in \(T(x)\) to fuzzy sets in \(U\)
LINGUISTIC VARIABLES

\[ X = (\text{Age}, T(\text{Age}), U, G, M) \]

- Age: name of the variable \( X \)
- \( T(\text{Age}) \): \{young, very young, very very young, \ldots\}
- \( U \): \([0,100] \) universe of discourse
- \( G(\text{Age}) \): \( T^{i+1} = \{\text{young}\} \cup \{\text{very } T^i\} \)
- \( M(\text{young}) = \{(u, \mu_{\text{young}}(u)) \mid u \in [0,100]\} \)

\[ \mu_{\text{young}}(u) = \begin{cases} 1 & \text{if } u \in [0,25] \\ \left(1 + \frac{u - 25}{5}\right)^{-2} & \text{if } u \in [25,100] \end{cases} \]
LINGUISTIC VARIABLES

An example of a fuzzy linguistic variable and membership functions
LINGUISTIC VARIABLES

- **Fuzzy linguistic terms often consist of two parts:**
  1) **Fuzzy predicate (primary term):** expensive, old, rare, dangerous, good, etc.
  2) **Fuzzy modifier:** very, likely, almost impossible, extremely unlikely, etc. Makes a composite linguistic term out of the primary term.

The modifier is used to change the meaning of predicate and it can be grouped into the following two classes:

  a) **Fuzzy truth qualifier or fuzzy truth value:** quite true, very true, more or less true, mostly false, etc.

  b) **Fuzzy quantifier:** many, few, almost, all, usually, etc.
**Fuzzy Predicate**

- **Fuzzy predicate**
  - If the set defining the predicates of individual is a fuzzy set, the predicate is called a fuzzy predicate

**Example**
- “z is expensive.”
- “w is young.”
- The terms “expensive” and “young” are fuzzy terms. Therefore the sets “expensive(z)” and “young(w)” are fuzzy sets
**Fuzzy Predicate**

- When a fuzzy predicate “x is P” is given, we can interpret it in two ways:
  - P(x) is a fuzzy set. The membership degree of x in the set P is defined by the membership function $\mu_{P(x)}$.
  - $\mu_{P(x)}$ is the satisfactory degree of x for the property P. Therefore, the truth value of the fuzzy predicate is defined by the membership function:
    - Truth value = $\mu_{P(x)}$
**Fuzzy Modifier (Hedge)**

- A new term can be obtained when we add a modifier “very” to a primary term
  - \( \mu_{\text{very young}}(u) = (\mu_{\text{young}}(u))^2 \)
Fuzzy Truth Qualifier

Baldwin defined fuzzy truth qualifier terms in the universal set $V = \{v \mid v \in [0,1]\}$ as follows

$T = \{\text{true}, \text{very true}, \text{fairly true}, \text{absolutely true}, \ldots, \text{absolutely false}, \text{fairly false}, \text{false}\}$
Fuzzy Truth Qualifier

Consider a proposition P “20 is young”. Truth value of this proposition is 0.9.

Consider

P₁ = “20 is young is true”
P₂ = “20 is young is fairly true”
P₃ = “20 is young is very true”
P₄ = “20 is young is false”
**Fuzzy Truth Qualifier**

**Example:** Let's define membership functions “young” and “old” of linguistic variable “age”

\[
\mu_{\text{young}}(x) = \text{bell}(x, 20, 2, 0) = \frac{1}{1 + \left(\frac{x}{20}\right)^4}
\]

\[
\mu_{\text{old}}(x) = \text{bell}(x, 30, 3, 100) = \frac{1}{1 + \left(\frac{x-100}{30}\right)^6}
\]

Where \(x\) is the age of a person in the universe of discourse \([0, 100]\)
Fuzzy Truth Qualifier

(a) Primary Linguistic Values

(b) Composite Linguistic Values

X = age

Membership Grades

Young

Old

Young but Not Too Young

Not Young and Not Old

More or Less Old

Extremely Old

X = age

Membership Grades
SUMMARY

- **Fuzzy logic** is based on predicate logic, where predicates are fuzzy.
- **Linguistic variables** are variables taking linguistic values. LVs form the foundation of approximate reasoning in fuzzy logic.
- Fuzzy predicates can be formed from a **primary term** and a **fuzzy modifier** (hedge).